THREE-DIMENSIONAL CHAINS AND THE ASSOCIATED COLLINEATIONS IN SPACE.

By Hazel Hope MacGregor.

Introduction.

Analytically the classification of the collineations in a linear space of any number of dimensions into types according to their invariant figures is equivalent to the classification of the linear homogeneous transformations in this space. The problem is thus intimately connected with the theory of elementary divisors of Weierstrass and on this analytic basis the classification for a space of n-dimensions has been made by Segré.*

In this work the coefficients of the transformation are supposed to be any complex numbers, and two collineations are regarded as equivalent if, and only if, one can be transformed into the other by a collineation with complex coefficients. From this point of view is obtained the well-known classification of nonsingular projective transformations on a line, in a plane, and in space; namely, two types on a line, five types in a plane, and thirteen types in space. This classification has also been made synthetically by Professor H. B. Newson.†

If, however, the coefficients in the collineation are restricted to be real numbers and if two such collineations are regarded as equivalent if, and only if, one can be transformed into the other by a transformation with real coefficients, the number of types is increased. Thus in the one-dimensional case we have the three well-known types usually designated by the terms hyperbolic, elliptic, and parabolic. Here, furthermore, arises the important additional problem as to the conditions under which a collineation with complex coefficients can be transformed into one with real coefficients. In the one-dimensional case again, these conditions are well known, it being necessary and sufficient that the projective transformation on the complex variable leave a circle in the Argand plane of the complex variable invariant.

In a recent paper‡ Professor J. W. Young has considered these problems for the one-dimensional case from the point of view of Projective Geometry,

* C. Segre, Memorie della R. Accademia delle Scienze di Torino. 1885.

† H. B. Newson, "Types of Projective Transformation in the Plane and in Space." Kansas University Quarterly, vol. VI, pages 64–69.

‡ J. W. Young, "The Geometry of Chains on a Complex Line." *Annals of Mathematics*, Second Series, vol. 11, October, 1909, pp. 33–48. This paper will be referred to as (A).

the notion of a linear chain being fundamental. In a subsequent paper* by making use of the idea of the two-dimensional chain Professor Young considers these problems for the collineations of the plane. He finds in all six different types of collineations in a plane which leave two-dimensional chains invariant and which may therefore be represented with real coefficients.

It is my purpose in this paper to apply this principle of classification to the nonsingular collineations in a complex space of three dimensions. To do this it will be necessary to introduce the *spatial*, or *three-dimensional* chain; the latter may be described as any class of points, lines, and planes in space which may be obtained from the class of real points, lines, and planes by a projective collineation. After defining a three-chain synthetically and observing some of its fundamental properties (§ 1) I give (§ 2) the classification of the collineations in space which leave three-dimensional chains invariant and derive the necessary and sufficient conditions that a collineation be of this type. Any such collineation may be represented with real coefficients.

§ 1. Definitions and Fundamental Properties of Three-Dimensional Chains.

The definitions and fundamental properties of linear and planar chains are assumed known. All points, lines, and planes considered are in the same complex three-dimensional space.

Definitions.—A point is said to be *linearly related* to two planar chains $\mathfrak{C}_1^2$ and $\mathfrak{C}_2^2$ on distinct planes and having a linear chain $\mathfrak{C}$ in common, provided it is the intersection of two lines each of which joins a point of $\mathfrak{C}_1^2$ to a distinct point of $\mathfrak{C}_2^2$. A line (plane) is said to be *linearly related* to $\mathfrak{C}_1^2$ and $\mathfrak{C}_2^2$ if it contains two distinct (three non-collinear) points which are linearly related to $\mathfrak{C}_1^2$, $\mathfrak{C}_2^2$. The set of all points, lines, and planes linearly related to $\mathfrak{C}_1^2$ and $\mathfrak{C}_2^2$ is called the *three-dimensional chain,* or more briefly, the *three-chain* defined by $\mathfrak{C}_1^2$ and $\mathfrak{C}_2^2$. This definition is equivalent to the analytic definition suggested in the introduction.

Fundamental Properties of Three-Chains.

I. *Through any five points, no four of which are coplanar, passes one and only one three-chain.*

This follows at once from the fact that by the Fundamental Theorem of Projectivity a projective collineation in space is uniquely determined

* J. W. Young, "Two-Dimensional Chains and the Associated Collineations in a Complex Plane." Transactions of the American Mathematical Society, vol. XI, 1910, pp. 280–293. Referred to as (B).

by five pairs, AA', BB', etc., of homologous points, no four of either of the sets $AB \ldots$ or $A'B' \ldots$ being coplanar.

II. *Any class of points, lines, and planes, in space projective with a three-chain is a three-chain.*

The next five properties are among what may be termed the *internal properties* of a three-chain. They follow at once from the well-known properties of the real three-space.

III. *Any two coplanar lines of a three-chain meet in a point of the three-chain.*

IV. *Any line of a three-chain meets any plane (not containing the line) of the three-chain in a point of the three-chain.*

V. *Any two distinct planes of a three-chain meet in a line of the three-chain.*

VI. *Any line of a three-chain has the points of a linear chain in common with the three-chain, and only these.*

VII. *A plane of a three-chain has the points and lines of a planar chain in common with the three-chain, and only these.*

The following properties may be termed *external properties* of a three-chain.

VIII. *Any three-chain is met by a plane not of the three-chain in a linear chain.*

Proof.—Let $ax_1 + bx_2 + cx_3 + dx_4 = 0$ be any plane in the complex space, where $a = a' + ia''$, $b = b' + ib''$, and so on, the a', a'', b', b'', . . . being real numbers. Then if x_1, x_2, x_3, x_4, are the coördinates of a point of the real space we have

$$a'x_1 + b'x_2 + c'x_3 + d'x_4 = 0,$$
$$a''x_1 + b''x_2 + c''x_3 + d''x_4 = 0.$$

These two equations have real coefficients and represent real planes; their intersection is a real line. But $ax_1 + bx_2 + cx_3 + dx_4 = 0$ is the equation of a plane passing through the intersection of these two planes. Hence every plane contains a real line and so by projection every plane has a linear chain in common with any three-chain.

Definition.—Two points are said to be *conjugate* to each other with respect to a given three-chain if they are the points into which a pair of conjugate imaginary points are transformed when the real three-chain is transformed into the given three-chain. Any point of a three-chain is *conjugate* with itself. The given three-chain is said to be *about* the points. Two lines are said to be *conjugate* with respect to a given three-chain if they are so related that every point on the one is the conjugate of some point on the other, in which case the three-chain is said to be *about* the two lines. Two projectivities on conjugate lines are said to be *conjugate* with

respect to the three-chain if they are so related that any two conjugate points are homologous with two conjugate points.

IX. *Any line joining two points conjugate to each other with respect to a given three-chain is a line of the three-chain.*

It is sufficient to recall that any line joining two conjugate imaginary points of real space is a real line, as the theorem then follows at once.

X. *A line joining any two points is conjugate with respect to a three-chain to a line joining the points conjugate to the two given points.*

XI. *Through any point not in a given three-chain there is one and only one line of the three-chain.*

If there were more than one, the point would be a point of the three-chain, and there is one, since the line joining a point to its conjugate point with respect to the three-chain is a line of the three-chain.

§ 2. A Classification of the Collineations in Space with Reference to their Invariant Three-Chains.

In this classification of the collineations in space with reference to their invariant three-chains, I shall show first, under what geometric conditions each type of collineation, based on a classification according to the invariant figure, will leave a three-chain invariant,—thus subdividing the type; then I shall show what restrictions must be placed on the three-chain in order that it remain invariant under this subtype.

Two theorems of frequent application are the following:

If a point P of an invariant three-chain of a collineation is not on an invariant line, any invariant plane through P is a plane of the three-chain.

Let P be transformed by the collineation into P' and P' into P''. The points P, P', P'' are not collinear but are the points of an invariant three-chain, and in an invariant plane, and so determine it as a plane of the three-chain.

If an invariant three-chain contains an invariant point, it contains an invariant planar chain, since point and planar chain are dual notions.

In the following, the results and terminology of papers (A) and (B) cited in the introduction are assumed known.

Type [1, 1, 1, 1].—The invariant figure of Type [1, 1, 1, 1] consists of the vertices, edges, and faces of a tetrahedron. The two-dimensional transformation in each invariant plane leaves a triangle invariant. The one-dimensional transformation on each invariant line leaves two elements invariant. Analytically this collineation is characterized by the fact that its characteristic equation has four distinct roots. If we assume the invariant three-chain to be the real three-chain, the coefficients of the characteristic equation are all real numbers and we have three cases to consider

according as the roots of the characteristic equation are all real, two real and two imaginary, or all imaginary. So any invariant three-chain contains four, two, or no double points of the collineation.

Type [1, 1, 1, 1]h.—Suppose the invariant three-chain to contain the four double points A, B, C, D. Then the projectivities on the *six* invariant lines are all hyperbolic. For the invariant line AB containing two points of the three-chain contains an invariant linear chain of the three-chain through A and B (A, p. 42) and similarly for BC, CA, AD, BD, and CD. The invariant three-chain contains an invariant planar chain on each of the invariant planes, since it contains three invariant points on each such plane.

The necessary and sufficient condition that a collineation of Type [1, 1, 1, 1] leave invariant a three-chain containing A, B, C, and D, is that the two-dimensional transformations in two of the invariant planes be hyperbolic of Type [1, 1, 1] (B).

That it is sufficient follows from the fact that two invariant planar chains determine the three-chain as invariant.

Through any point in space, not in an invariant plane, there is under this type of collineation one and only one invariant three-chain (I)* (cf. Theorem II (B)).

Type [1, 1, 1, 1]h-e.—If the invariant three-chain contains only two of the double points of the collineation, B and C, then it contains an invariant linear chain containing B and C, so the projectivity on BC is hyperbolic (A). The points A and D are then conjugate with respect to the invariant three-chain, and the line AD joining them must be a line of the three-chain (IX), so that AD contains an invariant linear chain of the invariant three-chain which does not contain the double points A and D. Therefore the projectivity on AD is elliptic (A).

The invariant three-chain contains invariant planar chains on BAD and on CAD, since each contains a point and line of the invariant three-chain.

The necessary and sufficient condition that a collineation of Type [1, 1, 1, 1] leave invariant a three-chain containing B and C, but not containing A and D, is that the two-dimensional transformations in ADB and ADC be elliptic of Type [1, 1, 1] (B).

It is sufficient since two invariant planar chains determine an invariant three-chain.

Through any point P in space not in an invariant plane there is under a collineation of this type one, and only one, invariant three-chain containing B and C, and about A and D.

Let BP meet ADC in a point B' and let CB' meet AD in B''. C, B', B'' determine a unique planar chain about A, D, which is invariant. A similar

* Roman numerals in ( ) refer to the fundamental properties in § 1.

invariant planar chain is determined in BAD. These two planar chains determine uniquely the invariant three-chain.

Type [1, 1, 1, 1]e.—Here the invariant three-chain contains none of the double points of the collineation. A, D and B, C are two pairs of points conjugate with respect to the invariant three-chain and the projectivities on AD and BC are elliptic. But this condition alone is not sufficient to characterize the collineation. To do this it is necessary to consider the projectivities on a pair of conjugates lines, such as AB and CD. Let P be any point of the invariant three-chain not on an invariant plane. Pass a plane through P and CD meeting AB in X, and a plane through P and AB meeting CD in Y. These two planes, being conjugate with respect to the three-chain, meet in a line of the three-chain, so X and Y are conjugate with respect to the three-chain. Now if P is transformed into P' and X into X', then Y is transformed into Y', the conjugate of X'.

The necessary and sufficient condition that a collineation of Type [1, 1, 1, 1] leave invariant a three-chain $\mathfrak{C}^3$ about the points A, D and about the points B, C is that the projectivities on AD and BC be elliptic and the projectivities on AB and CD be conjugate with respect to $\mathfrak{C}^3$.

That this is a sufficient condition may be seen as follows. Let P be any point not on an invariant plane. Draw the line through P meeting BC and AD. These points of intersection will determine linear chains about B, C and about A, D and the two linear chains and P determine a three-chain. This three-chain $\mathfrak{C}^3$ is invariant under the collineation, for P is on a line joining a point S of $\mathfrak{C}^3$ on AD to a point T of $\mathfrak{C}^3$ on BC and also on a line joining a point X of AB to the conjugate of X with respect to $\mathfrak{C}^3$ on CD. The collineation will put S and T into the points S' and T' of $\mathfrak{C}^3$, and it will put X into X' and Y into Y', the conjugate of X' with respect to $\mathfrak{C}^3$. $X'Y'$ and $S'T'$ are lines of $\mathfrak{C}^3$, so their intersection is a point of $\mathfrak{C}^3$, and therefore $\mathfrak{C}^3$ is invariant.

Through any point of space not on an invariant plane there is one and only one invariant three-chain of this type.

Let O be any point besides P, not on an invariant plane, and let π_1 be the projectivity leaving A, B, C, D invariant and putting P into O, and let π be the original collineation. Now $\pi_1\pi\pi_1^{-1} = \pi$, since the two projectivities have the same invariant points. But $\pi_1\pi\pi_1^{-1}$ leaves the three-chain through O invariant; therefore π leaves it invariant. That there is only one such three-chain follows readily from the proof of the last theorem.

If a collineation of Type [1, 1, 1, 1] leaves a three-chain invariant which contains the four invariant points A, B, C, D, the collineation may conveniently be designated as *hyperbolic of Type [1, 1, 1, 1]*, or *Type [1, 1, 1, 1]h*; if it contains only two of the invariant points, as *hyperbolic-elliptic of Type*

[1, 1, 1, 1], or *Type [1, 1, 1, 1]h-e*; and if it contains no invariant points as *elliptic of Type [1, 1, 1, 1]*, or *Type [1, 1, 1, 1]e.**

Type [2, 1, 1].—The invariant figure defining a collineation of Type [2, 1, 1] consists of three noncollinear points A, B, C; four lines, the sides of the triangle ABC, and a line l, through A and not in the plane ABC; and three invariant planes determined by l, AB, and AC.

If the invariant three-chain is the three-chain of real points it must contain either one or three of the invariant points of the collineation. The characteristic equation has real coefficients and so at least one real root. By the principle of duality it must contain either one or three of the invariant planes through A.

Type [2, 1, 1]h.—If an invariant three-chain contains the three double points of the collineation the projectivities on AB, AC, and BC are hyperbolic and on l parabolic. Conversely, if the projectivities on AB, AC, and BC are hyperbolic any three-chain containing A, B, C, and an invariant linear chain on l is invariant, being determined by an invariant planar chain on ABC and an invariant linear chain on l.

Through any point P, not in an invariant plane, there passes one and only one three-chain invariant under a collineation of this type.

Pass a plane through P and BC. It cuts l in a point, thus determining an invariant linear chain on l. Pass a plane through P and l. It cuts BC in a point which determines an invariant linear chain on BC. These two linear chains with P determine a three-chain $\mathfrak{C}^3$ which is invariant. To prove this draw a line through P and a point of the linear chain on l. It will cut the plane ABC in a point of $\mathfrak{C}^3$, which with A and the linear chain on BC determines an invariant planar chain on ABC. The invariant linear chain on l and the invariant planar chain on ABC determine the invariant three-chain.

The necessary and sufficient condition that a collineation leave invariant a three-chain containing A, B, C is that the two-dimensional projectivity in the plane ABC be hyperbolic of Type [1, 1, 1].

A collineation satisfying the conditions of this theorem may be designated as hyperbolic of Type *[2, 1, 1]*.

The necessary and sufficient condition that a three-chain be invariant under a collineation of Type [2, 1, 1]h is that it contain A, B, C and an invariant linear chain on l.

Type [2, 1, 1]e.—The invariant three-chain contains only one of the double points of the collineation, namely A, corresponding to the double root of the characteristic equation.

* It should perhaps be noted that these types are not mutually exclusive. They overlap when one or more of the projectivities on invariant lines are involutoric.

Any invariant three-chain containing A, but not B and C, contains an invariant planar chain on the plane ABC and an invariant linear chain on l, each containing A.

Since B and C are conjugate with respect to the invariant three-chain $\mathfrak{C}^3$, BC is a line of $\mathfrak{C}^3$ and this line with A determines an invariant planar chain of $\mathfrak{C}^3$ in the plane ABC. So the projectivity in ABC is elliptic of Type [1, 1, 1] (B). $\mathfrak{C}^3$ also has an invariant linear chain on l, for the plane ABl has at least a linear chain in common with any three-chain (VIII). Assume that it has a linear chain in common with $\mathfrak{C}^3$, not on l or AB (the latter is clearly impossible), but on some other line l'. This line is transformed into another line l'' which will also have a linear chain in common with $\mathfrak{C}^3$ and ABl. Both $\mathfrak{C}^3$ and the plane ABl are invariant under the collineation and these two linear chains determine a planar chain on ABl and therefore $\mathfrak{C}^3$ contains an invariant planar chain on ABl, which is impossible. Therefore $\mathfrak{C}^3$ must contain a linear chain on l, and since the projectivity on l is parabolic, this chain contains A. Therefore

The necessary and sufficient condition that a collineation of Type [2, 1, 1] leave invariant a three-chain containing only one of the double points is that the collineation in the plane ABC be elliptic of Type [1, 1, 1].

We will designate this collineation as *elliptic of Type [2, 1, 1]* or *Type [2, 1, 1]e.*

Through any point P in space, not in an invariant plane, passes one and only one invariant three-chain of this type.

The proof is similar to that for the corresponding theorem under Type [2, 1, 1]*h*.

The necessary and sufficient condition that a three-chain be invariant under a collineation of Type [2, 1, 1]e is that it contain a planar chain containing A and about B and C, and an invariant linear chain on l.

Type [2, 2].—The invariant figure of this collineation consists of two points, A, B, two planes, ABl and ABm, and three non-coplanar lines, AB, l, m, of which l contains A and m contains B. The two-dimensional transformations in ABl and ABm are both of the second type (B). The one-dimensional transformations are parabolic on l and m and of the first type on AB (A). The characteristic equation has two double roots, so that we have two cases to consider. Either the invariant three-chain contains the two double points, A and B, or else A and B are conjugate with respect to the invariant three-chain.

Type [2, 2]h.—If the invariant three-chain contains A and B, it contains two invariant planar chains.

The necessary and sufficient condition that a collineation of Type [2, 2] leave a three-chain through A and B invariant is that the projectivity on AB be hyperbolic.

A collineation satisfying this condition we will call *hyperbolic of Type [2, 2]* or *Type [2, 2]h.*

The necessary and sufficient condition that a three-chain be invariant under a collineation of Type [2, 2]h is that it contain an invariant linear chain on l and m.

If a three-chain contains these two linear chains it has an invariant planar chain on both ABl and ABm. These two invariant planar chains intersect in an invariant linear chain on AB and determine an invariant three-chain. That it is a necessary condition follows from the fact that an invariant three-chain containing A and B must contain two invariant planar chains.

Through any point P in space, not in one of the invariant planes, there passes one and only one of these invariant three-chains.

Let P' be the point into which P is transformed. Then $PP'A$ cuts m in a point of the three-chain determining the invariant linear chain on m, and $PP'B$ cuts l in a point determining the invariant chain on l. If we pass a plane through PP' and a point of the chain on l, it will cut AB in a point of the three-chain and so determine the linear chain on AB.

Type [2, 2]e.—If the invariant three-chain does not contain A and B, it contains no invariant planar chain and no invariant linear chains on l and m.

The necessary and sufficient condition that a collineation of Type [2, 2] leave invariant a three-chain $\mathfrak{C}^3$ not containing the double points, A and B, is that the projectivity on AB be elliptic and the projectivities on the two double lines l and m be conjugate with respect to $\mathfrak{C}^3$.

A collineation satisfying the above conditions we will call *elliptic of Type [2, 2]* or *Type [2, 2]e.*

There is one and only one invariant three-chain through every point not on a double plane.

These theorems may be proved as the similar theorems concerning the elliptic collineations of Type [1, 1, 1, 1].

Type [3, 1].—In this collineation we have two invariant points A and B, two invariant lines AB and l, passing through A, and two invariant planes, β determined by AB and l and α containing l. The two-dimensional projectivity in the plane β is of the second type and that in α of the third type. Three of the roots of the characteristic equation coincide so that all the roots are real. Any invariant three-chain therefore contains A and B, and by the principal of duality a planar chain on α, and one on β.

The necessary and sufficient condition that there be an invariant three-chain under a collineation of Type [3, 1] is that the projectivity on AB be hyperbolic.

Through any point P, not in α or β, there is under this collineation one and only one invariant three-chain.

Type [4].—The collineation of this type has as its invariant figure a single invariant point A in a single invariant plane α, and a single invariant line l in the invariant plane and through the invariant point. The two-dimensional transformation in the invariant plane is of the third type, and the one-dimensional transformation along the invariant line is parabolic. Any invariant three-chain contains A, since the roots of the characteristic equation all coincide, and, by the principle of duality, an invariant planar chain on α.

Every collineation of Type [4] *leaves invariant every three-chain containing an invariant planar chain on* α, *and a pair of homologous points not in* α.

Let P and P' be a pair of homologous points not on α; PP' cuts α in a point Q not on l. Let Q be transformed into Q'. QQ' cuts l in a point S, and S is transformed into S'. Let $\mathfrak{C}^3$ be the three-chain defined by the planar chain $| QQ'S'A |$ and PP'. $\mathfrak{C}^3$ is invariant. For $| QQ'S'A |$ is invariant (B). Now $PP' = PQ$ is transformed into $P'Q'$. The linear chain of planes on l in $\mathfrak{C}^3$ is invariant, since it contains α and a pair of homologous planes. Hence the plane lP' is transformed into lP'', a plane of $\mathfrak{C}^3$. The intersection of P'' and $P'Q'$ is the point P'' into which P' is transformed and is a point of $\mathfrak{C}^3$. Therefore $\mathfrak{C}^3$ is invariant and it is the only invariant three-chain through P, since every such three-chain must contain P', Q, Q' and therefore the planar chain on α. This is sufficient to determine the three-chain.

Perspective Collineations in Space.

A perspective collineation is determined by a plane of invariant points α, a center A which is invariant, and a pair of homologous points. There are two types of perspective collineations, one called "the homology in space," where A is not in α, and the other called "the elation in space," where A is in α.

Type [(1, 1, 1) 1].—In the case of the homology the one-dimensional transformations on the invariant lines through A and in the invariant pencils of rays and planes whose vertices and axes are in α are all of the first type, leaving two elements invariant.

Every invariant three-chain must contain A *and a planar chain on* α.

Let $\mathfrak{C}^3$ be any invariant three-chain. Let RR', PP', QQ' be any three pairs of homologous points of $\mathfrak{C}^3$ which are not all coplanar. Every pair of homologous points is collinear with A; therefore A is a point of $\mathfrak{C}^3$ (III). The dual argument proves that $\mathfrak{C}^3$ contains a planar chain on α.

A homology leaves invariant a three-chain if and only if the projectivity on a line through the center is hyperbolic.

Let $\mathfrak{C}^2$ be any planar chain on α and B be a point of $\mathfrak{C}^2$. Let P be any

point of the line BA, except B and A, and let the three-chain defined by $\mathfrak{C}^2$, A and P be denoted by $\mathfrak{C}^3$. Then $\mathfrak{C}^3$ is invariant, if and only if the projectivity on AP is hyperbolic. For then $|APB|$ would be invariant and P', the point into which P is transformed, would be in $\mathfrak{C}^3$. Since $\mathfrak{C}^3$ is determined by $\mathfrak{C}^2$, A and P, this proves also that $\mathfrak{C}^3$ is invariant.

Type [(2, 1, 1)].—In the case of the elations the one-dimensional transformations along all invariant lines through A and in all the invariant pencils of lines and planes whose vertices and axes are in α are parabolic.

By methods similar to those used under Type [(1, 1, 1), 1] it is easily proved that

The necessary and sufficient condition that a three-chain be invariant under a space elation is that it contain A, a planar chain on α, and an invariant linear chain on a line through A not in α. Every collineation of this type without restriction leaves a three-chain invariant.

The results for the remaining types can readily be obtained by the methods used hitherto. It seems desirable, therefore, to omit all proofs in the sequel.

Type [(1, 1), 1, 1].—The invariant figure of this type consists of all the points on a line AD, and two points, B and C, such that A, B, C, D, are not coplanar. The two-dimensional projectivities in the planes BAD and CAD are homologies, while those in the other invariant planes are of the first type.

Types [(1, 1), 1, 1]**h and** [(1, 1), 1, 1]**e**.—*If an invariant three-chain $\mathfrak{C}^3$ contains B and C, the two-dimensional projectivity in any plane through BC and a point of $\mathfrak{C}^3$ not on BC must be hyperbolic of Type [1, 1, 1]; if $\mathfrak{C}^3$ does not contain B and C, the two-dimensional projectivity in BPC must be elliptic of Type [1, 1, 1].*

If the invariant three-chain contains B and C the collineation may be called *hyperbolic of Type [(1, 1), 1, 1]*; if it does not contain these points it may be called *elliptic of Type [(1, 1), 1, 1]*.

Any three-chain containing a linear chain through B and C, or about B and C, and a linear chain on the line AD is invariant.

Type [2, (1, 1)].—The invariant figure of this collineation consists of all the points on a line BC and a point A, not on BC, and a line l through A not meeting BC. The two-dimensional transformation in the plane ABC is a homology and that in any invariant planes containing l is of Type [2, 1].

The necessary and sufficient condition that a collineation of Type [2, (1, 1)] leave a three-chain invariant is that the collineation in the plane ABC be hyperbolic. The necessary and sufficient condition that a three-chain be invariant under this collineation is that it contain a planar chain on ABC, containing A and the line BC, and an invariant linear chain on l.

Type [(1, 1), (1, 1)].—This type is determined by an invariant figure consisting of all the points on two nonintersecting lines, l and m, and of all lines and planes thereby determined. The one-dimensional transformations along the invariant lines are all of the first type, and the two-dimensional transformations in all the invariant planes are homologies. It can easily be proved that the projectivity on any invariant line which is not pointwise invarient, is projective with the projectivity on any other such line. So if the projectivity on one such invariant line is hyperbolic (elliptic) then the projectivity on every other such line is hyperbolic (elliptic).

The necessary and sufficient condition that a collineation of Type [(1, 1), (1, 1] *leave invariant a three-chain containing an invariant point is that the projectivity on an invariant line, not l or m, be hyperbolic, and that it leave invariant a three-chain not containing an invariant point, is that the projectivity on an invariant line, not l or m, be elliptic.*

Such a collineation we will call *hyperbolic of Type* [(1, 1), (1, 1)] if it contains an invariant point, otherwise we will call it *elliptic of Type* [(1, 1), (1, 1)].

The necessary and sufficient condition that a three-chain be invariant under the hyperbolic (elliptic) collineation of Type [(1, 1), (1, 1)] *is that it contain a linear chain through (about) the double points on each of two invariant lines, not l or m.*

Type [(2, 1), 1].—In this type all the points on a line AC, and a point B, not on AC, are invariant, and all the lines through A and lying in a plane π, containing AC but not B, are invariant. The two-dimensional transformation in the plane α, determined by B and AC, is a homology; that in π is an elation, and that in every plane through AB, except α, is of the second type.

A collineation of this type leaves a three-chain invariant, provided only the homology on α is hyperbolic and the necessary and sufficient condition on the three-chain that it remain invariant is that it contain A, B, and an invariant planar chain on π.

Type [(2, 2)].—In this type all the points on a line AC and all the planes of the pencil having this line as an axis are invariant. In each invariant plane all the lines through some point of the axis will be invariant and no two of these centers coincide. The two-dimensional projectivities in the invariant planes are all elations.

This type, without any restriction, leaves a three-chain invariant and the necessary and sufficient condition that a three-chain remain invariant is that it contain an invariant linear chain on each of two non-coplanar invariant lines.

Type [(3, 1)].—In this type the invariant figure consists of all the points

on a line AB, all the planes on an axis l through A, and all the lines through A in the plane β, determined by AB and l. The two-dimensional transformation in the plane β is an elation and that in all the other invariant planes is of the third type.

There is no restriction on the collineation in order that it leave a three-chain invariant. The necessary and sufficient condition that a three-chain be invariant is that it contain an invariant linear chain on l.

We have thus seen that the collineations in space that leave a three-chain invariant may be classified into nineteen subtypes under the thirteen general types. These nineteen subtypes are distributed as follows: three of Type [1, 1, 1, 1]; two of each of the Types [2, 1, 1], [2, 2] [(1, 1), 1, 1], and [(1, 1), (1, 1)]; and one of each of the other eight Types.

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